A CATEGORICAL APPROACH TO TURAEV'S HOPF GROUP-COALGEBRAS

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Abstract. We show that Turaev's group-coalgebras and Hopf group-coalgebras are coalgebras and Hopf algebras in a symmetric monoidal category, which we call the Turaev category. A similar result holds for group-algebras and Hopf group-algebras. As an application, we give an alternative approach to Virelizier's version of the Fundamental Theorem for Hopf algebras. We introduce Yetter-Drinfeld modules over Hopf group-coalgebras using the center construction.

Introduction

Group-coalgebras and Hopf group-coalgebras appeared in the work of Turaev [5] on homotopy quantum field theories. In the case where the underlying group G is trivial, we recover the classical coalgebras and Hopf algebras. A purely algebraic study of Hopf group-coalgebras was initiated by Virelizier [6], and then continued by Zunino [10, 11] and Wang [7, 8, 9]. It turns out that many of the classical results in Hopf algebra theory can be generalized to the group coalgebra setting. Virelizier gives a generalized version of the Fundamental Theorem for Hopf algebras, and introduces G-integrals; Zunino introduces Yetter-Drinfeld modules, the Drinfeld double, and a generalization of the center construction of a monoidal category; Wang introduces Doi-Hopf modules, entwined modules and coalgebra Galois theory for Hopf group-coalgebras, and he proves a version of Maschke's Theorem.

In contrast with classical Hopf algebras, the definition of a Hopf group-coalgebra is not selfdual. In fact, there exist the dual notions of group-algebra and Hopf group-algebra. The dual of a group-coalgebra is a group-algebra, and the converse property holds under some finiteness assumptions.

In this paper, we propose an alternative approach: we will introduce a symmetric monoidal category $\mathcal{T}_k$, called the Turaev category, and show that coalgebras (resp. Hopf algebras) in $\mathcal{T}_k$ are precisely group-coalgebras (resp. Hopf group-coalgebras). The objects of $\mathcal{T}_k$ are k-vector spaces (or k-modules, if k is a commutative ring), indexed by a set X. The morphisms are defined in such a way that we have a strongly monoidal forgetful functor to the opposite of the category of sets. We can define a second monoidal category $\mathcal{Z}_k$, called the Zunino category, with the same objects, but differently defined morphisms, such that the algebras (resp. Hopf algebras) in $\mathcal{Z}_k$ are the group-algebras (resp. Hopf group-algebras). The forgetful functor is now a strongly monoidal functor to the category of sets, and this will explain the lack of selfduality in the definition of Hopf group-coalgebras.

Some of the results of the above cited papers can now be viewed from the perspective of symmetric monoidal categories. We will discuss two examples. In Section 3, we

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will show that Virelizier's Fundamental Theorem can be viewed as a special case of Takeuchi's version of the Fundamental Theorem in braided monoidal categories [4]. In Section 4, we will compute the center of the category of modules over a Hopf group-coalgebra, which will lead to the introduction of Yetter-Drinfeld modules.

1. Preliminary results

1.1. **Monoidal categories.** Recall from e.g. [1] or [2] that a monoidal category (or a tensor category) consists of a triple $\mathcal{C} = (\mathcal{C}, \otimes, I)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, and I is an object in $\mathcal{C}$, together with natural isomorphisms

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

$$V \otimes I \cong V \cong I \otimes V$$

for all $U, V, W \in \mathcal{C}$, satisfying some appropriate coherence conditions. In all our examples, the associativity constraint is the obvious one, so we will identify $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$, and we will just write $U \otimes V \otimes W$. In a similar way, we identify $I \otimes V$, V and $V \otimes I$.
A braided monoidal category is a monoidal category together with a natural isomorphism $\tau_{V,W} : V \otimes W \to W \otimes V$ called the braiding or commutativity constraint, satisfying some appropriate coherence conditions. The category is called symmetric if $\tau_{W,V}$ is the inverse of $\tau_{V,W}$, for all V, W in $\mathcal{C}$.
An algebra (or monoid) in a monoidal category $\mathcal{C}$ is a triple (A, m, η), with A an object in $\mathcal{C}$, and $m : A \otimes A \to A$ and $\eta : I \to A$ morphisms in $\mathcal{C}$ such that

$$m \circ (A \otimes m) = m \circ (m \otimes A), \; m \circ (\eta \otimes A) = m \circ (A \otimes \eta) = A,$$

where A denotes the identity morphism of A. A right A-module is a couple (M, ψ), with $M \in \mathcal{C}$ and $\psi : M \otimes A \to M$ such that

$$\psi \circ (\psi \otimes A) = \psi \circ (M \otimes m) \text{ and } \psi \circ M \otimes \eta = M$$

The category of right A-modules will be denoted by $\mathcal{M}_A$.
The tensor product of two algebras A and B in a braided monoidal category is again an algebra, with multiplication

$$m_{A \otimes B} = (m_A \otimes m_B) \circ (A \otimes \tau_{B,A} \otimes B).$$

If $\mathcal{C}$ is a (braided) monoidal category, then $\mathcal{C}^{\rm op}$ is also a (braided) monoidal category, with the same tensor product and unit, and with the original constraints replaced by their inverses. A coalgebra in $\mathcal{C}$ is an algebra in $\mathcal{C}^{\rm op}$.
If (C, Δ, ε) is a coalgebra, and A is an algebra, then $\mathrm{Hom}_{\mathcal{C}}(C, A)$ is a semigroup, with multiplication (the convolution) given by

$$f * g = m \circ (f \otimes g) \circ \Delta$$

The unit for the convolution is $\eta \circ \varepsilon$.
A bialgebra in a braided monoidal category $\mathcal{C}$ is a fivetuple $H = (H, m, \eta, \Delta, \varepsilon)$ such that (H, m, η, Δ) is an algebra and (H, Δ, ε) is a coalgebra, and such that η and m are are coalgebra maps, or, equivalently, δ and ε are algebra maps. If the identity on H has an inverse S in $\mathrm{Hom}_{\mathcal{C}}(H, H)$, then $H = (H, m, \eta, \Delta, \varepsilon, S)$ is called a Hopf algebra in $\mathcal{C}$. S is called the antipode.

Example 1.1. Let k be a commutative ring, and consider the category $\mathcal{M}_k$ of k-modules and k-linear maps. Then $(\mathcal{M}_k, \otimes_k, k)$ is a symmetric monoidal category. A algebra in this category is a usual k-algebra. The same is true for a coalgebra, a bialgebra and a Hopf algebra. A module over an algebra A is a usual A-module, and the same is true for a comodule over a coalgebra.

Example 1.2. Let $\underline{\underline{\mathrm{Sets}}}$ be the category of sets and functions; $X \times Y$ is the cartesian product of two sets X and Y; fix a singleton $\{*\}$. Then $(\underline{\underline{\mathrm{Sets}}}, \times, \{*\})$ is a symmetric monoidal category. An algebra in $\underline{\underline{\mathrm{Sets}}}$ is a semigroup.

The following result is folklore, but we give details, as it will be useful in the sequel.

Lemma 1.3. *Let X be a set. There exists a unique comultiplication Δ and counit ε on X making (X, Δ, ε) a coalgebra in $\underline{\underline{\mathrm{Sets}}}$. Δ and ε are the following maps:*

$$\Delta:\ X \to X \times X,\ \Delta(x) = (x, x)\ ;\ \varepsilon:\ X \to \{*\},\ \varepsilon(x) = *. \tag{1}$$

A bialgebra $(X, m, \eta, \Delta, \varepsilon)$ in $\underline{\underline{\mathrm{Sets}}}$ is of the following form: (X, m, η) is a semigroup, and Δ and ε are given by (1).
Finally, the category of Hopf algebras in $\underline{\underline{\mathrm{Sets}}}$ is isomorphic to the category of groups.

Proof. Let (X, Δ, ε) be a coalgebra in $\underline{\underline{\mathrm{Sets}}}$. Then ε is the unique map from X to $\{*\}$. Take $x \in X$, and assume that $\Delta(x) = (y, z) \in X \times X$. Then

$$(x, *) = (y, \varepsilon(z)) \text{ and } (*, x) = (\varepsilon(y), z)$$

hence $x = y = z$. Conversely, it is a straightforward to check that (X, Δ, ε) is a coalgebra in $\underline{\underline{\mathrm{Sets}}}$. If (X, m, η) is a semigroup, then the maps Δ and ε are semigroup maps, hence $(X, m, \eta, \Delta, \varepsilon)$ is a bialgebra in $\underline{\underline{\mathrm{Sets}}}$.
Let X be a set (hence a coalgebra), and G a semi-group (hence an algebra). The convolution on $\mathrm{Map}(X, G)$ is then just pointwise multiplation: for $f, g:\ X \to G$, we have

$$(f * g)(x) = f(x)g(x)$$

The unit element of the convolution is $\eta \circ \varepsilon:\ X \to G$, mapping every $x \in X$ to the unit element 1 of G.
Now let $(X, m, \eta, \Delta, \varepsilon, s)$ be a Hopf algebra in $\underline{\underline{\mathrm{Sets}}}$. Then $s * X = X * s = \eta \circ \varepsilon$, hence for all $x \in X$:

$$(m \circ (s, X) \circ \Delta)(x) = (m \circ (X, s) \circ \Delta)(x) = 1$$

or

$$s(x)x = xs(x) = 1$$

so $s(x) = x^{-1}$, and X is a group. Conversely, if X is a group, then we can give it a bialgebra structure, and $s(x) = x^{-1}$ is an antipode. $\square$

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor $F:\ \mathcal{C} \to \mathcal{D}$ is a functor together with a natural transformation

$$u_{V,W}:\ F(V) \otimes F(W) \to F(V \otimes W)$$

and a morphism

$$J \to F(I)$$

compatible with the constraints. If this natural transformation and this morphism are isomorphisms, then we call the monoidal functor strong. A monoidal functor sends algebras to algebras, and a strong monoidal functor also sends coalgebras

to coalgebras. A braided monoidal functor between two braided mocategories is a monoidal functor between the monoidal categories that is compatible with the braidings, i.e.

$$u_{W,V} \circ t_{F(V),F(W)} = F(t_{V,W}) \circ u_{V,W}$$

for all $V, W \in \mathcal{C}$. A braided strong monoidal functor sends bialgebras to bialgebras and Hopf algebras to Hopf algebras.

A strong monoidal functor sends algebras, coalgebras, bialgebras and Hopf algebras to algebras, coalgebras, bialgebras and Hopf algebras. An example of a strong monoidal functor is the functor $F: \underline{\underline{\mathrm{Sets}}} \to \mathcal{M}_k$ sending a set X to the free k-module with basis X. F sends a group G (a Hopf algebra in $\underline{\underline{\mathrm{Sets}}}$) to the group algebra kG, which is a Hopf algebra in $\mathcal{M}_k$.

If A is an algebra in $\mathcal{C}$, then a right A-module M is an object $M \in \mathcal{C}$ together with a morphism $\psi: M \otimes A \to M$ such that

$$\psi \circ (\psi \otimes A) = \psi \circ (A \otimes \mu) \text{ and } \psi \circ (M \otimes \eta) = M$$

A morphism f between two right A-modules M and N is called right A-linear if $\psi \circ (f \otimes A) = f \circ \psi$. The category of right A-modules is denoted by $\mathcal{C}_A$.
The category of comodules $\mathcal{C}^C$ over a coalgebra C is defined in a dual fashion. If H is a bialgebra in a braided monoidal category $\mathcal{C}$, a (right) Hopf module is an object $M \in \mathcal{C}$ together with $\psi: M \otimes H \to M$ and $\rho: M \to M \otimes H$ such that

$$\rho \circ \psi = (\psi \otimes \mu) \circ t_{23} \circ (\rho \otimes \Delta) \tag{2}$$

1.2. **The center of a monoidal category.** Let $\mathcal{C}$ be a monoidal category. The weak left center $\mathcal{W}_l(\mathcal{C})$ is the category with the following objects and morphisms. An object is a couple (V, s), with $V \in \mathcal{C}$ and $s: V \otimes - \to - \otimes V$ a natural transformation between functors $\mathcal{C} \to \mathcal{C}$, satisfying the following condition, for all $X, Y \in \mathcal{C}$:

$$(X \otimes s_Y) \circ (s_X \otimes Y) = s_{X \otimes Y}, \tag{3}$$

and such that s_I is the composition of the natural isomorphisms $V \otimes I \cong V \cong I \otimes V$.
A morphism between (V, s) and (V', s') consists of $f: V \to V'$ in $\mathcal{C}$ such that

$$(X \otimes f) \circ s_X = s'_X \circ (f \otimes X)$$

$\mathcal{W}_l(\mathcal{C})$ is a prebraided monoidal category. The unit is (I, I), and the tensor product is

$$(V, s) \otimes (V', s') = (V \otimes V', u)$$

with

$$u_X = s_X \otimes V' \circ V \otimes s'_X \tag{4}$$

The prebraiding c on $\mathcal{W}_l(\mathcal{C})$ is given by

$$c_{V,V'} = s_{V'}: (V, s) \otimes (V', s') \to (V', s') \otimes (V, s). \tag{5}$$

The left center $\mathcal{Z}_l(\mathcal{C})$ is the full subcategory of $\mathcal{W}_l(\mathcal{C})$ consisting of objects (V, s) with s a natural isomorphism. $\mathcal{Z}_l(\mathcal{C})$ is a braided monoidal category. For detail in the case where $\mathcal{C}$ is a strict monoidal category, we refer to [1, Theorem XIII.4.2]. The results remain valid in the case of an arbitrary monoidal category, since every monoidal category is equivalent to a strict one.

1.3. **G-coalgebras and Hopf G-coalgebras.** From [5], we recall the following definitions.

Definition 1.4. Let G be a group, en k a commutative ring. A G-coalgebra is a family of k-modules $C = (C_g)_{g\in G}$ indexed by the group G, together with a family of linear maps

$$\Delta = (\Delta_{g,h} :\ C_{gh} \to C_g \otimes C_h)_{g,h\in G}$$

and a linear map $\varepsilon :\ C_1 \to k$ such that the following conditions hold, for all $g, h, k \in G$:

$$(\Delta_{g,h} \otimes C_k) \circ \Delta_{gh,k} = (C_g \otimes \Delta_{h,k}) \circ \Delta_{g,hk}, \tag{6}$$

$$(C_g \otimes \varepsilon) \circ \Delta_{g,1} = C_g = (\varepsilon \otimes C_g) \circ \Delta_{1,g}, \tag{7}$$

In [6], the following Sweedler-type notation for comultiplication is introduced: for $c \in C_{gh}$, we write

$$\Delta_{g,h}(c) = c_{(1,g)} \otimes c_{(2,h)}.$$

For $c \in C_{ghk}$, we will write, using the coassociativity,

$$((\Delta_{g,h} \otimes C_k) \circ \Delta_{gh,k})(c) = ((C_g \otimes \Delta_{h,k}) \circ \Delta_{g,hk})(c) = c_{(1,g)} \otimes c_{(2,h)} \otimes c_{(3,k)}.$$

Definition 1.5. We use the notation from Definition 1.4. A semi-Hopf G-coalgebra is a family of k-algebras $H = (H_g)_{g\in G}$ indexed by the group G, together with families of linear maps Δ and ε such that H is a G-coalgebra, and such that ε and $\Delta_{g,h}$ are algebra maps.
A Hopf G-coalgebra is a semi-Hopf G-coalgebra together with a family of maps

$$S = \{S_g :\ H_g^{-1} \to H_g \mid g \in G\}$$

such that

$$\mu_g \circ (S_g \otimes H_g) \circ \Delta_{g^{-1},g} = \eta_g \circ \varepsilon = \mu_g \circ (H_g \otimes S_g) \circ \Delta_{g^{-1},g}, \tag{8}$$

for all $g \in G$, where μ_g and η_g are the multiplication and counit maps on H_g.

Observe that a G-coalgebra and a semi-Hopf G-coalgebra can also be defined in the case where G is just a monoid.

1.4. **G-algebras and Hopf G-algebras.** The definition of G-coalgebra and Hopf G-coalgebra can be dualized. This was already remarked in [5]; a formal definition has been presented in [10].

Definition 1.6. Let G be a group. A G-algebra consists of a set of k-modules $A = (A)_{g\in G}$ together with maps

$$\mu_{g,h} :\ A_g \otimes A_h \to A_{gh} \text{ and } \eta :\ k \to A_1$$

such that

$$\mu_{gh,k} \circ (\mu_{g,h} \otimes A_k) = \mu_{g,hk} \circ (A_g \otimes \mu_{h,k}) \tag{9}$$

and

$$\mu_{g,1} \circ (A_g \otimes \eta) = A_g = \mu_{1,g} \circ (\eta \otimes A_g) \tag{10}$$

Definition 1.7. Let G be a group. A G-algebra $H = (H)_{g\in G}$ is called a semi-Hopf G-algebra if every H_g is a k-coalgebra in such a way that $\mu_{g,h}$ and η are k-coalgebra maps.

Definition 1.8. Let G be a group. A Hopf G-algebra is a semi-Hopf G-algebra $H = (H)_{g\in G}$ together with maps $S_g:\ H_g \to H_{g^{-1}}$ such that

$$\mu_{g^{-1},g} \circ (S_g \otimes H_g) \circ \Delta_g = \mu_{g,g^{-1}} \circ (H_g \otimes S_g) \circ \Delta_g = \eta \circ \varepsilon_g \tag{11}$$

Note that *pi*-algebras and semi-Hopf G-algebras can be defined in the situation where G is only a semigroup.

2. The Turaev and Zunino categories

In Definition 1.4, we fix a group G. The central idea is to replace G by a variable set.

2.1. **The Turaev category.**

Definition 2.1. Let k be a commutative ring. A Turaev k-module is a couple $\underline{M} = (X, M)$, where X is a set, and $M = (M_x)_{x\in X}$ is a family of k-modules indexed by X. A morphism between two T-modules (X, M) and (Y, N) is a couple $\underline{\varphi} = (f, \varphi)$, where $f:\ Y \to X$ is a function, and $\varphi = (\varphi_y:\ M_{f(y)} \to N_y)_{y\in Y}$ is a family of linear maps indexed by Y. The composition of $\underline{\varphi}:\ \underline{M} \to \underline{N}$ and $\underline{\psi}:\ \underline{N} \to \underline{P} = (Z, P)$ is defined as follows:

$$\underline{\psi} \circ \underline{\varphi} = (f \circ g, (\psi_z \circ \varphi_{g(z)})_{z\in Z})$$

The category of Turaev k-modules is denoted by $\mathcal{T}_k$.

We will use the following notation for the composition of morphisms:

$$\begin{array}{ccccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} & \xrightarrow{\psi} & \underline{P} \\ X & \xleftarrow{f} & Y & \xleftarrow{g} & Z \\ M_{f(g(z))} & \xrightarrow{\varphi_{g(z)}} & N_{g(z)} & \xrightarrow{\psi_z} & P_z \end{array} \tag{12}$$

The category of T-modules is a symmetric monoidal category. The tensor product of (X, M) and (Y, N) is given by

$$(X, M) \otimes (Y, M) = (X \times Y, (M_x \otimes N_y)_{(x,y)\in X\times Y}),$$

and the unit object is $(\{*\}, k)$. The symmetry

$$\underline{\tau} = (t, \tau):\ (X \times Y, (M_x \otimes N_y)_{(x,y)\in X\times Y}) \to (Y \times X, (N_y \otimes M_x)_{(y,x)\in Y\times X})$$

is defined as follows: $t:\ Y \times X \to X \times Y$ is the switch map, and $\tau_{t(y,x)}:\ M_x \otimes N_y \to N_y \otimes M_X$ is also the switch map. We have two strong monoidal functors

$$F':\ \mathcal{M}_k \to \mathcal{T}_k,\ F'(M) = (\{*\}, M)$$

and

$$F:\ \mathcal{T}_k \to \underline{\underline{\mathrm{Sets}}}^{\mathrm{op}},\ F(X, M) = X$$

Proposition 2.2. *If (G, C) is a coalgebra in $\mathcal{T}_k$, then G is a monoid and C is a G-coalgebra. Conversely, if C is a G-coalgebra, then (G, C) is a coalgebra in $\mathcal{T}_k$.*

Proof. Let $\underline{C} = (\underline{C} = (G, C), \underline{\Delta} = (m, \Delta), \underline{\varepsilon} = (i, \varepsilon))$ be a coalgebra in $\mathcal{T}_k$. Then $F(\underline{C}) = (G, m, i)$ is a coalgebra in $\underline{\underline{\mathrm{Sets}}}^{\mathrm{op}}$, hence a monoid. The counit and comultiplication are

$$\begin{array}{ccc} \underline{C} & \xrightarrow{\varepsilon} & \underline{k} \\ G & \xleftarrow{i} & \{*\} \\ C_1 = C_{i(e)} & \xrightarrow{\varepsilon} & k \end{array} \quad \text{and} \quad \begin{array}{ccc} \underline{C} & \xrightarrow{\underline{\Delta}} & \underline{C} \otimes \underline{C} \\ G & \xleftarrow{m} & G \times G \\ C_{gh} = C_{m(g,h)} & \xrightarrow{\Delta_{g,h}} & C_g \otimes C_h \end{array} \tag{13}$$

We have

$$\begin{array}{ccccc}
\underline{C} & \xrightarrow{\underline{\Delta}} & \underline{C}\otimes\underline{C} & \xrightarrow{\underline{C}\otimes\underline{\Delta}} & \underline{C}\otimes\underline{C}\otimes\underline{C}\\
G & \xleftarrow{m} & G\times G & \xleftarrow{(G,m)} & G\times G\times G\\
C_{ghk} & \xrightarrow{\Delta_{g,hk}} & C_g\otimes C_{hk} & \xrightarrow{C_g\otimes\Delta_{h,k}} & C_g\otimes C_h\otimes C_k
\end{array}$$

and

$$\begin{array}{ccccc}
\underline{C} & \xrightarrow{\underline{\Delta}} & \underline{C}\otimes\underline{C} & \xrightarrow{\underline{\Delta}\otimes\underline{C}} & \underline{C}\otimes\underline{C}\otimes\underline{C}\\
G & \xleftarrow{m} & G\times G & \xleftarrow{(m,G)} & G\times G\times G\\
C_{ghk} & \xrightarrow{\Delta_{gh,k}} & C_{gh}\otimes C_k & \xrightarrow{\Delta_{g,h}\otimes C_k} & C_g\otimes C_h\otimes C_k
\end{array}$$

hence $\underline{\Delta}$ is coassociative if and only if (6) holds, for all $g,h,k\in G$. In a similar way

$$\begin{array}{ccccc}
\underline{C} & \xrightarrow{\underline{\Delta}} & \underline{C}\otimes\underline{C} & \xrightarrow{\underline{\varepsilon}\otimes\underline{C}} & \underline{C}\\
G & \xleftarrow{m} & G\times G & \xleftarrow{(i,G)} & G\\
C_g & \xrightarrow{\Delta_{1,g}} & C_1\otimes C_g & \xrightarrow{\varepsilon\otimes C_g} & C_g
\end{array}$$

and consequently $(\underline{\varepsilon}\otimes\underline{C})\circ\underline{\Delta}=\underline{C}$ if and only if the second equality in (7) holds, for all $g\in G$. In a similar way, $(\underline{C}\otimes\underline{\varepsilon})\circ\underline{\Delta}=\underline{C}$ if and only if the first equality in (7) holds. Hence C is a G-coalgebra.
Conversely, given a G-coalgebra C, we define a coalgebra structure on (G,C) by (13). □

Proposition 2.3. *An algebra (X,A) in $\mathcal{T}_k$ consists of a set X and a family of k-algebras $A=(A_x)_{x\in X}$ indexed by X. A map (f,φ) in $\mathcal{T}_k$ between two algebras (X,A) and (Y,B) is an algebra map if and only if every $\varphi_{f(y)}:\ A_{f(y)}\to B_y$ is a k-algebra map.*

Proof. Take an algebra $\underline{A}=((X,A),\underline{\mu}=(\delta,\mu),\underline{\eta}=(e,\eta))$ in $\mathcal{T}_k$. Then $F(\underline{A})$ is an algebra in $\underline{\underline{\mathrm{Sets}}}^{\mathrm{op}}$, hence a coalgebra in $\underline{\underline{\mathrm{Sets}}}$. It follows from Lemma 1.3 that X is an arbitray set, and that $\delta(x)=(x,x)$ and $e(x)=*$. The multiplication and unit are morphisms

$$\begin{array}{ccc}
\underline{A}\otimes\underline{A} & \xrightarrow{\underline{\mu}} & \underline{A}\\
X\times X & \xleftarrow{\delta} & X\\
A_x\otimes A_x & \xrightarrow{\mu_x} & A_x
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\underline{k} & \xrightarrow{\underline{\eta}} & \underline{A}\\
\{*\} & \xleftarrow{e} & X\\
k & \xrightarrow{\eta_x} & A_x
\end{array}$$

Let us compute

$$\begin{array}{ccccc}
\underline{A}\otimes\underline{A}\otimes\underline{A} & \xrightarrow{\underline{A}\otimes\underline{\mu}} & \underline{A}\otimes\underline{A} & \xrightarrow{\underline{\mu}} & \underline{A}\\
X\times X\times X & \xleftarrow{(X,\delta)} & X\times X & \xleftarrow{\delta} & X\\
A_x\otimes A_x\otimes A_x & \xrightarrow{A_x\otimes\mu_x} & A_x\otimes A_x & \xrightarrow{\mu_x} & A_x
\end{array}$$

In a similar way, we compute $\underline{\mu}\circ(\underline{A}\otimes\underline{\mu})$, and it follows that $\underline{\mu}$ is associative if and only if every μ_x is associative.
We also have

$$\begin{array}{ccccc}
\underline{A} & \xrightarrow{\underline{A}\otimes\underline{\eta}} & \underline{A}\otimes\underline{A} & \xrightarrow{\underline{\mu}} & \underline{A}\\
X & \xleftarrow{(X,e)} & X\times X & \xleftarrow{\delta} & X\\
A_x & \xrightarrow{A_x\otimes\eta_x} & A_x\otimes A_x & \xrightarrow{\mu_x} & A_x
\end{array}$$

hence $\underline{\mu}\circ(\underline{A}\otimes\underline{\eta})=\underline{A}$ if and only if $\mu_x\circ(A_x\otimes\eta_x)=A_x$ for every $x\in X$. The same is true for the left unit property, and our result follows. The final statement is straightforward, and is left to the reader. □

Proposition 2.4. *If (G,H) is a bialgebra in $\mathcal{T}_k$, then G is a monoid and H is a semi-Hopf G-coalgebra. Conversely, if H is a semi-Hopf G-coalgebra, then (G,H) is a bialgebra in $\mathcal{T}_k$.*

Proof. Let (G,H) be a bialgebra in $\mathcal{T}_k$. We know from Propositions 2.2 and 2.3 that G is a monoid, H is a G-coalgebra, and every H_g is a k-algebra. It follows from the final statement in Proposition 2.4 that $\underline{\varepsilon}$ and $\underline{\Delta}$ are algebra maps if and only if ε and $\Delta_{g,h}$ are k-algebra maps, for all $g,h\in G$. □

Proposition 2.5. *If (G,H) is a Hopf algebra in $\mathcal{T}_k$, then G is a group and H is a Hopf G-coalgebra. Conversely, if H is a Hopf G-coalgebra, then (G,H) is a Hopf algebra in $\mathcal{T}_k$.*

Proof. Let $\underline{H}=(G,H)$ be a Hopf algebra in $\mathcal{T}_k$. Then $F(\underline{H})=G$ is a Hopf algebra in $\underline{\underline{\mathrm{Sets}}}^{\mathrm{op}}$, and also in $\underline{\underline{\mathrm{Sets}}}$, since the definition of a Hopf algebra in a category is selfdual. Thus G is a group. Let $s:\ G\to G$, $s(g)=g^{-1}$, and consider a map $\underline{S}=(s,S):\ \underline{H}\to\underline{H}$ in $\mathcal{T}_k$. We compute the convolution $\underline{S}*\underline{H}$.

$$\begin{array}{ccccccc}
\underline{H} & \xrightarrow{\underline{\Delta}} & \underline{H}\otimes\underline{H} & \xrightarrow{\underline{S\otimes H}} & \underline{H}\otimes\underline{H} & \xrightarrow{\underline{\mu}} & \underline{H}\\
G & \xleftarrow{m} & G\times G & \xleftarrow{(s,G)} & G\times G & \xleftarrow{\delta} & G\\
H_1 & \xrightarrow{\Delta_{g^{-1},g}} & H_{g^{-1}}\otimes H_g & \xrightarrow{S_g\otimes H_g} & H_g\otimes H_g & \xrightarrow{\mu_g} & H_g
\end{array}$$

We also compute

$$\begin{array}{ccccc}
\underline{H} & \xrightarrow{\underline{\varepsilon}} & \underline{k} & \xrightarrow{\underline{\eta}} & \underline{H}\\
G & \xleftarrow{1} & \{*\} & \xleftarrow{e} & G\\
H_1 & \xrightarrow{\varepsilon} & k & \xrightarrow{\eta_g} & H_g
\end{array}$$

So $\underline{S}*\underline{H}=\underline{\eta}\circ\underline{\varepsilon}$ if and only if

$$\mu_g\circ(S_g\otimes H_g)\circ\Delta_{g^{-1},g}=\eta_g\circ\varepsilon$$

In a similar way, $\underline{H}*\underline{S}=\underline{\eta}\circ\underline{\varepsilon}$ if and oly if

$$\mu_g\circ(H_g\otimes S_g)\circ\Delta_{g^{-1},g}=\eta_g\circ\varepsilon,$$

and our result follows, in view of (8) □

The next result will turn out to be important when we discuss the Fundamental Theorem for Hopf G-coalgebras. But let us first recall that the category $\underline{\underline{\mathrm{Sets}}}$ has coequalizers (see e.g. [2, p. 65]). For two maps $f,g:\ Y\to X$, one considers the smallest equivalence relation $\sim$ on X containing

$$\{(f(y),g(y))\mid y\in Y\}$$

The coequalizer of f and g is the natural surjection $x\to\overline{X}=X/\sim$. The class in $\overline{X}$ represented by $x\in X$ will be denoted by $\overline{x}$, and observe that $f^{-1}(\overline{x})=g^{-1}(\overline{x})$.

Proposition 2.6. *The category $\mathcal{T}_k$ has equalizers and coequalizers.*

Proof. Take two morphisms in $\mathcal{T}_k$:

$$\begin{array}{ccc}
\underline{M} & \xrightarrow{\underline{\varphi}} & \underline{N}\\
X & \xleftarrow{f} & Y\\
M_{f(y)} & \xrightarrow{\varphi_y} & N_y
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\underline{M} & \xrightarrow{\underline{\psi}} & \underline{N}\\
X & \xleftarrow{g} & Y\\
M_{g(y)} & \xrightarrow{\psi_y} & N_y
\end{array}$$

Take $\overline{x}\in\overline{X}$, and put

$$\overline{M}_{\overline{x}}=\{(m_x)\in\prod_{x\in\overline{x}}M_x\mid\forall y\in f^{-1}(\overline{x})=g^{-1}(\overline{x}):\ \varphi_y(m_{f(y)})=\psi_y(m_{g(y)})\}$$

Set $\underline{\overline{M}} = (\overline{X}, \overline{M} = (\overline{M}_{\overline{x}})_{\overline{x}\in\overline{X}})$. We claim that the equalizer of $\overline{\varphi}$ and $\overline{\psi}$ is

$$\begin{array}{ccc} \underline{\overline{M}} & \longrightarrow & \underline{M} \\ \overline{X} & \longleftarrow & X \\ \overline{M}_{\overline{x}} & \longrightarrow & M_x \end{array}$$

where $\overline{M}_{\overline{x}} \to M_x$ is the restriction to $\overline{M}_{\overline{x}}$ of the projection $\prod_{x\in\overline{x}} M_x \to M_x$. Consider a morphism $\underline{\pi}:\ \underline{P} \to \underline{M}$ in $\mathcal{T}_k$ such that $\underline{\varphi} \circ \underline{\pi} = \underline{\psi} \circ \underline{\pi}$. Thus

$$\begin{array}{ccccc} \underline{P} & \xrightarrow{\underline{\pi}} & \underline{M} & \xrightarrow{\underline{\varphi}} & \underline{N} \\ Z & \xleftarrow{h} & X & \xleftarrow{f} & Y \\ P_{h(f(y))} & \xrightarrow{\pi_{f(y)}} & M_{f(y)} & \xrightarrow{\varphi_y} & N_y \end{array} \quad \text{and} \quad \begin{array}{ccccc} \underline{P} & \xrightarrow{\underline{\pi}} & \underline{M} & \xrightarrow{\underline{\psi}} & \underline{N} \\ Z & \xleftarrow{h} & X & \xleftarrow{g} & Y \\ P_{h(g(y))} & \xrightarrow{\pi_{g(y)}} & M_{g(y)} & \xrightarrow{\psi_y} & N_y \end{array}$$

We now define a morphism $\underline{\overline{\pi}}:\ \underline{P} \to \underline{\overline{M}}$. The map

$$\overline{h}:\ \overline{X} \to Z,\ \overline{h}(\overline{x}) = h(x)$$

is well-defined. Take $\overline{x} \in \overline{X}$. For all $x' \in \overline{x}$, we have that $h(x') = h(x)$, and we have a map $\pi_{x'}:\ P_{x'} \to M_x$. Now define

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\underline{\overline{\pi}}} & \underline{\overline{M}} \\ Z & \xleftarrow{\overline{h}} & \overline{X} \\ P_{\overline{h}(\overline{x})} & \xrightarrow{\pi_{\underline{x}}} & \underline{M}_{\underline{x}} \end{array}$$

as follows: we know that $P_{\overline{h}(\overline{x})} = P_{h(x')}$, and we define

$$\underline{\pi}_{\underline{x}}(p) = \Big(\pi_{x'}(p)\Big)_{x'\in\underline{x}}.$$

Suppose that $x = f(y)$ and $x' = g(y')$. Then

$$\varphi_y(\pi_{f(y)}(p)) = \psi_y(\pi_{g(y)}(p)),$$

hence $\underline{\pi}_{\underline{x}}(p) \in \overline{M}_{\underline{x}}$, as needed.
Let us now prove that $\mathcal{T}_k$ has coequalizers. Let

$$U = \{u \in Y \mid f(u) = g(u)\}$$

For $u \in U$, we have maps

$$\varphi_u, \psi_u:\ M_{f(u)} = M_{g(u)} \to N_u$$

We now put $P_u = \mathrm{Coker}\,(\varphi_u - \psi_u)$, en $\underline{P} = (Z, (P_u)_{u\in U})$.

$$\begin{array}{ccc} \underline{N} & \longrightarrow & \underline{P} \\ Y & \longleftarrow & U \\ M_{f(y)} = M_{g(y)} & \longrightarrow & P_u \end{array}$$

is a morphism in $\mathcal{Z}_k$, and is the coequalizer of $\underline{\varphi}$ and $\underline{\psi}$. □

2.2. **The Zunino category.** Let k be a commutative ring. The objects of the Zunino category $\mathcal{Z}_k$ are the same as the objects of $\mathcal{T}_k$: a set X together with a family of k-modules $(M_x)_{x\in X}$ indexed by X. A morphism between $\underline{M} = (X, M = (M_x)_{x\in X})$ and $\underline{N} = (T, N = (N_y)_{y\in Y})$ consists of a map $f:\ X \to Y$ and a family of k-module homomorphisms $\varphi_x:\ M_x \to N_{f(x)}$. We use the following notation: $\underline{\varphi} = (f, \varphi):\ \underline{M} \to \underline{N}$, or

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\underline{\varphi}} & \underline{N} \\ X & \xrightarrow{f} & Y \\ M_x & \xrightarrow{\varphi_x} & N_{f(x)} \end{array}$$

The category $\mathcal{Z}_k$ is a symmetric monoidal category; the tensor product and unit object are the same as the ones on $\mathcal{T}_k$. We also have strong monoidal functors

$$F' : \mathcal{M}_k \to \mathcal{Z}_k, \ F'(M) = (\{*\}, M)$$
$$F : \mathcal{Z}_k \to \underline{\underline{\text{Sets}}}, \ F(X, M) = X$$

The proof of the following results is similar to the corresponding proof of Propositions 2.2, 2.3, 2.4 and 2.5. We leave the details to the reader.

Proposition 2.7. *Assume that (G, A) is an algebra in $\mathcal{Z}_k$. Then G is a monoid, and A is a G-algebra, in the sense of Definition 1.6. Conversely, if A is a G-algebra, then (G, A) is an algebra in $\mathcal{Z}_k$.*

Proposition 2.8. *A coalgebra in $\mathcal{Z}_k$ consists of a family of coalgebras C_x indexed by a set X.*

Proposition 2.9. *Assume that (G, H) is a bialgebra in $\mathcal{Z}_k$. Then G is a monoid, and H is a semi-Hopf G-algebra, in the sense of Definition 1.7. Conversely, if H is a semi-Hopf G-algebra, then (G, H) is a bialgebra in $\mathcal{Z}_k$.*

Proposition 2.10. *Assume that (G, H) is a Hopf algebra in $\mathcal{Z}_k$. Then G is a group, and H is a Hopf G-algebra, in the sense of Definition 1.8. Conversely, if H is a Hopf G-algebra, then (G, H) is a Hopf algebra in $\mathcal{Z}_k$.*

Let k be a commutative ring. We have braided monoidal contravariant functors

$$^* : \mathcal{T}_k \to \mathcal{Z}_k \text{ and } ^* : \mathcal{Z}_k \to \mathcal{T}_k$$

For $\underline{M} = (X, (M_x)_{x \in X} \in \mathcal{T}_k$, we let

$$\underline{M}^* = (X, (M_x^*)_{x \in X}.$$

A similar definition holds for $\underline{M} = (X, (M_x)_{x \in X} \in \mathcal{Z}_k$. Let us illustrate that the functors are monoidal: for $\underline{M} = \underline{N} \in \mathcal{T}_k$, we have the following morphism in $\mathcal{Z}_k$:

$$\begin{array}{ccc} \underline{M}^* \otimes \underline{N}^* & \longrightarrow & (\underline{M} \otimes \underline{N})^* \\ X \times Y & \xleftarrow{=} & X \times Y \\ M_x^* \otimes N_y^* & \longrightarrow & (M_x \otimes N_y)^* \end{array}$$

Consequently, if $\underline{C}$ is a coalgebra in $\mathcal{T}_k$, then $\underline{C}^*$ is an algebra in $\mathcal{Z}_k$ (in fact the duality functor is a covariant functor from $\mathcal{T}_k^{\text{op}}$ to $\mathcal{Z}_k$, hence it sends algebras to algebras, see Section 1.1).

We call $\underline{M} = (X, (M_x)_{x \in X}) \in \mathcal{T}_k$ (resp. in $\mathcal{Z}_k$) *finite* if M_x is finitely generated and projective as a k-module. If $\underline{M}$ and $\underline{N}$ are finite, then $M_x^* \otimes N_y^* \cong (M_x \otimes N_y)^*$, for every $x \in X$ and $y \in Y$. Hence the duality functor induces a strong monoidal functor between the full subcategories $\mathcal{T}_k^f$ and $\mathcal{Z}_k^f$ consisting of finite objects, and the two duality functors establish a pair of inverse equivalences.

Thus the dual of a finite algebra in $\mathcal{Z}_k$ is a coalgebra in $\mathcal{T}_k$, and the dual of a bialgebra or Hopf algebra in $\mathcal{T}_k$ (resp. $\mathcal{Z}_k$) is a bialgebra or Hopf algebra in $\mathcal{Z}_k$ (resp. $\mathcal{T}_k$). These facts are known (see [8, 10]).

Proposition 2.11. *The category $\mathcal{Z}_k$ has equalizers and coequalizers.*

Proof. Take two morphisms in $\mathcal{Z}_k$:

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} \\ X & \xrightarrow{f} & Y \\ M_x & \xrightarrow{\varphi_x} & N_{f(x)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \underline{M} & \xrightarrow{\psi} & \underline{N} \\ X & \xrightarrow{g} & Y \\ M_x & \xrightarrow{\psi_x} & N_{g(y)} \end{array}$$

Let $Z = \mathrm{Ker}\,(f,g) = \{x \in X \mid f(x) = g(x)\}$, and, for each $x \in Z$, $P_x = \mathrm{Ker}\,(\varphi_x - \psi_x)$. Then the equalizer of $\underline{\varphi}$ and $\underline{\psi}$ is

$$\begin{array}{ccc} \underline{P} & \longrightarrow & \underline{M} \\ Z & \xrightarrow{\subset} & X \\ P_x & \xrightarrow{\subset} & M_x \end{array}$$

Let $\overline{Y} = \mathrm{Coker}\,(f,g)$. Take $\overline{y} \in \overline{Y}$ and $y' \in \overline{y}$. We have a map

$$i_{y'}:\ N_{y'} \to \prod_{y \in \overline{y}} N_y,\ (i_{y'}(n))_y = \begin{cases} n & \text{if} \quad y = y' \\ 0 & \text{if} \quad y \neq y' \end{cases}$$

On $\prod\limits_{y \in \overline{y}} N_y$, we consider the smallest equivalence relation $\sim$ containing

$$\{\Big((i_{f(x)} \circ \varphi_x)(m), (i_{g(x)} \circ \psi_x)(m)\Big) \mid x \in f^{-1}(\overline{y}) = g^{-1}(\overline{y}),\ m \in M_x\},$$

and we define

$$\overline{N}_{\overline{y}} = \prod_{y \in \overline{y}} N_y / \sim .$$

The coequalizer of $\underline{\varphi}$ and $\underline{\psi}$ is

$$\begin{array}{ccc} \underline{N} & \longrightarrow & \underline{\overline{N}} \\ Y & \longrightarrow & \overline{Y} \\ N_y & \xrightarrow{p_y} & \overline{N}_{\overline{y}} \end{array}$$

where p_y is the composition of i_y and the canonical projection

$$\prod_{y \in \overline{y}} N_y \to \prod_{y \in \overline{y}} N_y / \sim .$$

$\square$

3. The Fundamental Theorem

It is known, see e.g. [4, Theorem 3.4] that the Fundamental Theorem of Hopf modules, as stated in Sweedler's book [3], can be formulated in any braided monoidal category with equalizers. If the category has coequalizers, then we have a second version of the Fundamental Theorem, since a Hopf algebra is also a Hopf algebra in the opposite category. Since $\mathcal{Z}_k$ and $\mathcal{T}_k$ have equalizers and coequalizers, we have two versions of the Fundamental Theorem in both categories. The aim of this Section is to make this explicit, and to derive Virelizier's version of the Fundamental Theorem from it (see [6, Theorem 2.7]). First we look at what is happening in the category of sets.

3.1. **Hopf modules in the category of sets.** Let G be a set, and consider the corresponding coalgebra (G, δ, e) in $\underline{\underline{\mathrm{Sets}}}$. Let (X, ρ) be a right G-comodule. It is easy to show that $\rho(x) = (x, f(x))$, with $f:\ X \to G$ an arbitrary map. Thus a right G-comodule consists of a set X together with a map $f:\ X \to G$.
Now take a semigroup G, that is an algebra in $\underline{\underline{\mathrm{Sets}}}$. A right G-module is a right G-set X, this is a set X together with a G-action such that $x1 = x$ and $(xg)h = x(gh)$, for all $x \in X$ and $g, h \in G$.

A semigroup G is also a bialgebra in $\underline{\underline{\mathrm{Sets}}}$, so we can consider right Hopf modules over G. These are right G-sets X together with a map $f:\ X\to G$ satisfying

$$f(xg)=f(x)g \tag{14}$$

for all $x\in X$ and $g\in G$.

Let X be a right G-Hopf module. The coinvariants $X^{\mathrm{co}G}$ of X are defined as the equalizer of (X,f) and (X,η), that is

$$X^{\mathrm{co}G}=\{x\in X\mid f(x)=1\}$$

The Fundamental Theorem then takes the following form.

Theorem 3.1. *Let G be a group, and X be a right G-Hopf module. Then the map*

$$\phi:\ X^{\mathrm{co}G}\times G\to X,\ \phi(x,g)=xg$$

is bijective.

Proof. This is a special case of [4, Theorem 3.4], but it can also be verified directly. The inverse of ϕ is given by

$$\phi^{-1}(x)=(xf(x)^{-1},f(x))$$

□

A group G is also a Hopf algebra in $\underline{\underline{Sets}}^{\mathrm{op}}$. Let X be a G-Hopf module, and consider X_G, the coequalizer of

$$X\times G\overset{\mathrm{action}}{\longrightarrow}\text{ and }X\times G\overset{(X,e)}{\longrightarrow}X$$

In X_G, x and xg are identified, which means that X_G consists of the orbits under the G-action. The orbit containing x will be denoted by $\overline{x}$. The second version of the Fundamental Theorem is now the following.

Theorem 3.2. *Let G be a group, and X be a right G-Hopf module. Then the map*

$$\psi:\ X\to X\times G,\ \psi(x)=(\overline{x},f(x))$$

is bijective.

Proof. Again a special case of [4, Theorem 3.4]; but we can also verify directly that ψ^{-1} is given by

$$\psi^{-1}(\overline{x},g)=xf(x)^{-1}g$$

□

3.2. **Modules, comodules and Hopf modules in $\mathcal{T}_k$.** . Let $\underline{C}=(G,C)$ be a coalgebra in $\mathcal{T}_k$. We can consider the category $\mathcal{T}^{\underline{C}}$ of right $\underline{C}$-comodules and right $\underline{C}$-colinear maps. An object $\underline{M}=(X,M)\in\mathcal{T}^{\underline{C}}$ consists of a right G-set X and a set of k-modules indexed by X, together with a coaction

$$\underline{\rho}:\ \underline{M}\to\underline{M}\otimes\underline{C}$$

in $\mathcal{T}_k$ such that

$$(\underline{M}\otimes\underline{\varepsilon})\circ\underline{\rho}=\underline{M}\text{ and }(\underline{\rho}\otimes\underline{\Delta})\circ\underline{\rho}=(\underline{\rho}\otimes\underline{C})\circ\underline{\rho}$$

Explicitely,

$$\begin{array}{ccc}\underline{M}&\overset{\underline{\rho}}{\longrightarrow}&\underline{M}\otimes\underline{C}\\ X&\longleftarrow&X\times G\\ M_{xg}&\overset{\rho_{x,g}}{\longrightarrow}&M_x\otimes C_g\end{array}$$

satisfies

$$(M_x \otimes \varepsilon) \circ \rho_{x,1} = M_x \text{ and } (\rho_{x,g} \otimes C_h) \circ \rho_{xg,h} = (M_x \otimes \Delta_{g,h}) \circ \rho_{x,gh}$$

Note that our definition is more general than the one in [6], where G is a group, and $X = G$.

Example 3.3. Let X be a right G-set, and M a k-module. Let $\rho_g: M \to M \otimes C_g$ be maps such that

$$(\rho_g \otimes C_h) \circ \rho_h = (M \otimes \Delta_{g,h}) \circ \rho_{gh} \text{ and } (M \otimes \varepsilon) \circ \rho_1 = M.$$

Then $(\{*\}, M)$ is a right $\overline{C}$-comodule. For all $x \in X$, let $M_x = M$ as a k-module, and $\underline{M} = \{X, (M_x)_{x\in X} \in \mathcal{T}_k$. Define $\underline{\rho}: \underline{M} \to \underline{M} \otimes \underline{C}$ as follows:

$$\rho_{x,g} = \rho_g: M_{xg} = M \to M_x \otimes C_g = M \otimes C_g.$$

Then $\underline{M}$ is a right $\underline{C}$-comodule.

Now let $\underline{A} = (Y, A)$ be an algebra in $\mathcal{T}_k$. A right $\underline{A}$-module $\underline{M} = (X, M)$ consists of a set of modules M_x indexed by the set X, and a map $f: X \to Y$ such that M_x is a right $A_{f(x)}$-module.
If $\underline{H}$ is a bialgebra in $\mathcal{T}_k$, then we can consider (right) Hopf modules. Let $\underline{M} = (X, M)$ be such a Hopf module. Then $\underline{M}$ is a right $\underline{H}$-module and a right $\underline{H}$-comodule, as above. The map $f: X \to G$ has to be right G-linear, and (2) has to be satisfied, explicitely

$$(\psi_x \circ M_g) \circ (M_x \otimes t_{g,f(x)} \otimes M_g) \circ (\rho_{x,g} \otimes \Delta_{f(x),g}) = \rho_{x,g} \circ \psi_{xg}$$

for all $x \in X$ and $g \in G$. Here $\psi_x: M_x \otimes H_{f(x)} \to M_x$ is the right $H_{f(x)}$-action on M_x, and $t_{g,f(x)}: H_g \otimes H_{f(x)} \to H_{f(x)} \otimes H_g$ is the switch map.

3.3. **The Fundamental Theorem in the category $\mathcal{T}_k$.** Let $\underline{H}$ be a Hopf algebra in $\mathcal{T}_k$, and consider a right Hopf module $\underline{M}$. The module of coinvariants $\underline{M}^{\mathrm{co}\underline{H}}$ is by definition the equalizer of $\underline{\rho}$ and $\underline{M} \otimes \underline{\eta}$. $\underline{M}^{\mathrm{co}\underline{H}}$ can be described as follows:

$$\underline{M}^{\mathrm{co}\underline{H}} = (X_G, (M_{\overline{x}})_{\overline{x}\in X_G})$$

with

$$M_{\overline{x}}^{\mathrm{co}\underline{H}} = \{(m_y) \in \prod_{y\in\overline{x}} M_y \mid \rho_{y,g}(m_{xg}) = m_x \otimes 1_g, \text{ for all } y \in \overline{x} \text{ and } g \in G\}$$

Theorem 3.4. (Fundamental Theorem) *Let $\underline{H}$ be a Hopf algebra in $\mathcal{T}_k$, and $\underline{M}$ a Hopf module. Then we have an isomorphism of Hopf modules*

$$\begin{array}{ccc} \underline{M}^{\mathrm{co}\underline{H}} \otimes \underline{H} & \xrightarrow{\underline{\phi}} & \underline{M} \\ X_G \times G & \xleftarrow{\varphi} & X \\ M_{\overline{x}} \otimes H_{f(x)} & \xrightarrow{\phi_{x,g}} & M_x \end{array}$$

with $\varphi(x) = (\overline{x}, f(x))$ and $\phi_{x,g}((m_y)_{y\in\overline{x}}) \otimes h_{f(x)}) = m_x h_{f(x)}$.

Proof. xxx This is a direct consequence of Theorem 3.2. Let us give the explicit formula for $\underline{\phi}^{-1}$. First observe that the two compositions below coincide:

$$\underline{M} \xrightarrow{\underline{\rho}} \underline{M} \otimes \underline{H} \xrightarrow{\underline{M}\otimes\underline{S}} \underline{M} \otimes \underline{H} \xrightarrow{\underline{\psi}} \underline{M} \underset{\underline{M}\otimes\underline{H}}{\overset{\underline{\rho}}{\rightrightarrows}} \underline{M} \otimes \underline{H}$$

Thus we have a map $\underline{R}:\ \underline{M}\to \underline{M}^{\mathrm{co}\underline{H}}$. According to the proof of Proposition 2.6, $\underline{R}$ can be described explicitely as follows:

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\underline{R}} & \underline{M}^{\mathrm{co}\underline{H}} \\ X & \xleftarrow{r} & X_G \\ M_{xf(x)^{-1}} & \xrightarrow{R_{\overline{x}}} & M_{\overline{x}}^{\mathrm{co}\underline{H}} \end{array}$$

with $r(\overline{x})=xf(x)^{-1}$. For $x'\in\overline{x}$, we consider $m\in M_{x'f(x')^{-1}}=M_{xf(x)^{-1}}$, and

$$\rho_{x',f(x')^{-1}}(m)=m_{[0,x']}\otimes m_{[1,f(x')^{-1}]}$$

Then

$$R_{\overline{x}}(m)=\big(m_{[0,x']}S_{f(x')}(m_{[1,f(x')^{-1}]})\big)_{x'\in X}$$

and $\underline{\phi}^{-1}$ is the composition $(\underline{R}\otimes\underline{H})\circ\underline{\rho}$, that is

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\underline{\phi}^{-1}} & \underline{M}^{\mathrm{co}\underline{H}}\otimes\underline{H} \\ X & \xleftarrow{\varphi^{-1}} & X_G\times G \\ M_{xf(x)^{-1}g} & \xrightarrow{\phi^{-1}_{\overline{x},g}} & M_{\overline{x}}^{\mathrm{co}\underline{H}}\otimes H_g \end{array}$$

with

$$\varphi^{-1}(\overline{x},g)=xf(x)^{-1}g$$

and

$$\phi^{-1}_{\overline{x},g}(m)=\big(m_{[0,x']}S_{f(x')}(m_{[1,f(x')^{-1}]})\big)_{x'\in X}\otimes m_{[2,g]}$$

□

Remark 3.5. Let us consider the particular situation where $X=G$ and $f:\ G\to G$ is the identity map. In this situation, a version of the Fundamental Theorem appears in [6], so let us compare this to our Theorem 3.4. $X_G=\{\overline{e}\}$, the G-action on G has only one orbit, and

$$\underline{M}_{\overline{e}}^{\mathrm{co}\underline{H}}=\{(m_g)\in\prod_{g\in G}M_g\mid \rho_{g,h}(m_{gh})=m_g\otimes 1_h,\ \text{for all } g,h\in G\}$$

Virelizier then considers the image of the projection on the g-component and calls this $M_g^{\mathrm{co}\underline{H}}$. Then he takes $M^{\mathrm{co}\underline{H}}$ to be the product of all the $M_g^{\mathrm{co}\underline{H}}$. He then defines the tensor product in a different way (see [6, 2.6]), namely

$$(M^{\mathrm{co}\underline{H}}\otimes\underline{H})_g=M_g^{\mathrm{co}\underline{H}}\otimes H_g$$

while we have

$$(\underline{M}^{\mathrm{co}\underline{H}}\otimes\underline{H})_g=\underline{M}_{\overline{e}}^{\mathrm{co}\underline{H}}\otimes H_g$$

We will show next that these two tensor products are isomorphic, so that Virelizier's Fundamental Theorem is a special case of Theorem 3.4 (and a fortiori of [4, Theorem 3.4]).

Proposition 3.6. *Let k be a field. With notation as in Remark 3.5, we have that*

$$M_g^{\mathrm{co}\underline{H}}\cong\underline{M}_{\overline{e}}^{\mathrm{co}\underline{H}},$$

in other words, the projection $p_g:\ \underline{M}_{\overline{e}}^{\mathrm{co}\underline{H}}\to M_g^{\mathrm{co}\underline{H}}$ is injective.

Proof. Take $(m_h)_{h\in G} \in \underline{M}_{\overline{e}}^{\mathrm{co}\underline{H}}$. We have to show that the full string is completely determined by one single entry m_g. This can be seen as follows.

$$\rho_{h,h^{-1}g}(m_g) = m_h \otimes 1_{h^{-1}g} \in M_h \otimes H_{h^{-1}g}$$

Since k is a field, the subspace of $H_{h^{-1}g}$ spanned by $1_{h^{-1}g}$ has a complement in $H_{h^{-1}g}$, so we have a projection $p\ H_{h^{-1}g} \to k$, mapping $1_{h^{-1}g}$ to 1. Applying p to the second tensor factor, we see that

$$m_h = (M_h \otimes p)(\rho_{h,h^{-1}g}(m_g)).$$

□

4. Yetter-Drinfeld modules and the center construction

Let H be a Hopf group-coalgebra. Yetter-Drinfeld modules over H have been studied in [10, 11, 7]. The aim of this Section is to recover (a generalized version) of Yetter-Drinfeld modules using the centre construction. This result is inspired by the classical result that, for a Hopf algebra H, the centre of the category of H-modules is isomorphic to the category of Yetter-Drinfeld modules. First we need some set theory. The following definition goes back to Whitehead.

Definition 4.1. Let G be a monoid. A right crossed G-set is a right G-set V together with a map $\nu :\ V \to G$ such that

$$g\nu(v \cdot g) = \nu(v)g,$$

for all $g \in G$ and $v \in V$. A morphism between two right crossed G-sets (V,ν) and (W,ω) is a morphism of G-sets $f :\ V \to W$ such that $\nu = \omega \circ f$. The category of right crossed G-sets will be denoted by $\mathcal{X}_G^G$.

The category $\mathcal{S}_G$ of right G-sets is a monoidal category. The tensor product of two G-sets X and Y is the cartesian product, with the diagonal action. The unit is $\{*\}$ with the trivial G-action $* \cdot g = *$. The following result is classical.

Proposition 4.2. *Let G be a monoid. The left weak center $\mathcal{W}_l(\mathcal{S}_G)$ is isomorphic to the category of right crossed G-sets $\mathcal{X}_G^G$. If G is a group, then it is also isomorphic to the center.*

Proof. (sketch) Let $(V,s) \in \mathcal{W}_l(\mathcal{S}_G)$, and consider $s_G :\ V \times G \to G \times V$, and write

$$s_G(v,1) = (\nu(v), f(v)),$$

with $\nu :\ V \to G$, $f :\ V \to V$.
Take a right G-set X, and consider the morphism of G-sets $\varphi :\ G \to X$, $\varphi(g) = xg$. From the naturality of s, it follows that

$$s_X \circ (V,\varphi) = (\varphi, V) \circ s_G.$$

Applying this to $(v,1)$, we see that $s_X(v,x) = (x\nu(v), f(v))$. We then that

$$(*,v) = s_*(v,*) = (* \cdot \nu(v), f(v)) = (*, f(v)),$$

so it follows that $f = V$ is the identity map on V. We conclude that s is completely determined by ν:

$$s_X(v,x) = (x \cdot \nu(v), v). \tag{15}$$

Using the fact that s_G is right G-linear, we find

$$(\nu(v)g, v \cdot g) = s_G(v,1) \cdot 1 = s_G(v \cdot g, g) = (g\nu(v \cdot g), v \cdot g),$$

so $f(v)g = g\nu(v \cdot g)$, proving that (V, ν) is a right crossed G-set.
Conversely, if (V, ν) is a right crossed G-set, then we define $(V, s) \in \mathcal{W}_l(\mathcal{S}_G)$ using (15). $\square$

Let G be a set. As we have seen, there is a unique coalgebra structure on G in the category of sets. We can then consider the category $\mathcal{S}^G$ of right G-comodules. Its objects are couples (X, f), with X a set, and $f : X \to G$ a map. A morphism between (X, f) and (Y, g) is a map $\xi : X \to Y$ such that $g \circ \xi = f$. If G is a monoid, then $\mathcal{S}^G$ is a monoidal category. The tensor product is

$$(X, f) \times (Y, g) = (X \times Y, fg),$$

with $fg : X \times Y \to G$, $(fg)(x, y) = f(x)g(y)$. The unit is $(\{*\}, i)$, with $i(*) = 1$. For any set X, we can consider the right G-comodule $X_1 = (M, 1)$, with $1 : X \to G$ the constant map taking value 1. Then for every $X = (X, f) \in \mathcal{S}^G$, the map

$$\rho : X \to X_1 \times G, \ \rho(x) = (x, f(x)),$$

is a morphism in $\mathcal{S}^G$. This will be needed to prove the following result, which is perhaps not so well-known as Proposition 4.2.

Proposition 4.3. *Let G be a monoid. The left weak center $\mathcal{W}_l(\mathcal{S}^G)$ is isomorphic to the category of right crossed G-sets $\mathcal{X}_G^G$. If G is a group, then it is also isomorphic to the center.*

Proof. Take $((V, \nu), s) \in \mathcal{W}_l(\mathcal{S}^G)$. Take a set X, and fix $x \in X$. The map

$$g : \{*\} \to X_1, \ g(*) = x$$

is a morphism in $\mathcal{S}^G$. From the naturality of s, it follows that the following diagram is commutative

$$\begin{tikzcd} V \times \{*\} \arrow[r, "s_* = V"] \arrow[d, "(V, g)"'] & \{*\} \times V \arrow[d, "(g, V)"] \\ V \times X_1 \arrow[r, "s_{X_1}"] & X_1 \times V \end{tikzcd}$$

and it follows that $s_{X_1}(v, x) = (x, v)$.
$G = (G, G)$ is an object of $\mathcal{S}^G$, hence we can consider $s_G : V \times G \to G \times V$; let us denote

$$s_G(v, g) = (\gamma(v, g), v \cdot g),$$

for all $v \in G$ and $g \in G$. Since $((V, \nu), s) \in \mathcal{W}_l(\mathcal{S}^G)$, we have that

$$(X_1, s_G) \circ (s_{X_1}, G) = s_{X_1 \times G},$$

implying that

$$s_{X_1 \times G}(v, x, g) = (x, \gamma(v, g), v \cdot g).$$

From the naturality of s, we have the following commutative diagram

$$\begin{tikzcd} V \times X \arrow[r, "s_X"] \arrow[d, "(V, \rho)"'] & X \times V \arrow[d, "(\rho, V)"] \\ V \times X_1 \times G \arrow[r, "s_{X_1 \times G}"] & X_1 \times G \times V \end{tikzcd}$$

Take $(v,x) \in V \times M$, and write $s_M(v,x) = (y,w)$. The commutativity of the diagram implies that

$$(y, f(y), w) = s_{X_1\times G}(v, x, f(x)) = (x, \gamma(v, f(x)), v \cdot f(x)).$$

Looking at the first component, we see that $y = x$. In particular, taking $M = G$, it follows that $\gamma(v,g) = g$. Looking at the third component, we see that $w = v \cdot f(x)$, and we conclude that

$$s_X(v,x) = (x, v \cdot f(x)), \tag{16}$$

and, in particular,

$$s_G(v,g) = (g, v \cdot g).$$

s_G is a morphism in $\mathcal{S}^G$, hence $G\nu \circ s_G = \nu G$, and it follows that

$$\nu(v)g = g\nu(v \cdot g),$$

for all $g \in G$ and $v \in V$. From the fact that

$$s_{G\times G} = (G, s_V) \circ (s_V, G),$$

we conclude that $s_{G\times G}(v,(g,h)) = ((g,h),(v\cdot g)\cdot h)$. Since $m:\ G\times G \to G$ is a morphism in $\mathcal{S}^G$, we have the following commutative diagram from the naturality of s:

$$\begin{tikzcd}
V \times G \times G \arrow[r, "s_{G\times G}"] \arrow[d, "{(V,m)}"'] & G \times G \times V \arrow[d, "{(m,V)}"] \\
V \times G \arrow[r, "s_G"] & G \times V
\end{tikzcd}$$

and it follows that $(v\cdot g)\cdot h = v\cdot(gh)$. Finally, the constant map $1:\ \{*\} \to G$ is a morphism in $\mathcal{S}^G$, so we have a commutative diagram

$$\begin{tikzcd}
V \times \{*\} \arrow[r, "s_*"] \arrow[d, "{(V,1)}"'] & \{*\} \times V \arrow[d, "{(1,V)}"] \\
V \times G \arrow[r, "s_G"] & G \times V
\end{tikzcd}$$

from which we deduce that $v \cdot 1 = v$. Thus we have shown that (V,ν) is a right crossed G-module.

Conversely, given a right crossed G-module (V,ν), we define $((V,\nu),s) \in \mathcal{W}_l(\mathcal{S}^G)$ using (16). $\square$

Now let $\underline{H} = (G,(H_g)_{g\in G})$ be a semi-Hopf group coalgebra. In particular, this means that G is a monoid.

Definition 4.4. A right-right $\underline{H}$-Yetter-Drinfeld module consists of the following data:

- An object $\underline{M} = (V,(M_v)_{v\in V}) \in \mathcal{T}_k$;
- a right $\underline{H}$-comodule structure $\underline{\rho}:\ \underline{M} \to \underline{M} \otimes \underline{H}$; in particular, V is a right G-set;
- a right $\underline{H}$-module structure $\underline{\psi}:\ \underline{M} \otimes \underline{H} \to \underline{M}$; this implies that we have a map $\nu:\ V \to V \times G$ making $(V,f) \in \mathcal{S}^G$,

satisfying the following compatibility relations:

- V is a crossed right G-set;
- for all $v \in V$, $g \in G$, $m \in M_{v\cdot g}$ and $h \in H_{g\nu(v\cdot g)} = H_{\nu(v)g}$, we have

$$\big(mh_{(2,\nu(v\cdot g))}\big)_{[0,v]} \otimes h_{(1,g)}\big(mh_{(2,\nu(v\cdot g))}\big)_{[1,g]} = m_{[0,v]}h_{(1,\nu(v))} \otimes m_{[1,g]}h_{(2,g)}. \tag{17}$$

The category of right-right Yetter-Drinfeld modules and $\underline{H}$-linear $\underline{H}$-colinear maps is denoted by $\mathcal{YD}^{\underline{H}}_{\underline{H}}$.

The category of right $\underline{H}$-modules is a monoidal category. For $\underline{P} = (X, P), \underline{Q} = (Y, Q) \in \mathcal{T}_{\underline{H}}$, we have the following right $\underline{H}$-coaction on $\underline{P} \otimes \underline{Q}$:

$$\begin{array}{ccc} \underline{P} \otimes \underline{Q} \otimes \underline{H} & \xrightarrow{\psi} & \underline{P} \otimes \underline{Q} \\ X \times Y \times G & \xleftarrow{X\times Y\times fg} & X \times Y \\ P_x \otimes Q_y \otimes H_{f(x)g(y)} & \xrightarrow{\psi_{x,y}} & P_x \otimes Q_y, \end{array}$$

with

$$\psi_{x,y}(p \otimes q \otimes h) = ph_{(1,f(x))} \otimes qh_{(1,g(y))},$$

for all $p \in P_x$, $q \in Q_x$ and $h \in H_{f(x)g(y)}$. We can therefore consider the centre and the weak centre of $\mathcal{T}_{\underline{H}}$, and the main result of this Section is now the following.

Theorem 4.5. *Let $\underline{H}$ be a semi-Hopf group coalgebra. Then we have an isomorphism of categories*

$$\mathcal{W}_r(\mathcal{T}_{\underline{H}}) \cong \mathcal{YD}^{\underline{H}}_{\underline{H}}.$$

If $\underline{H}$ is a Hopf group coalgebra, then the weak centre is equal to the centre.

Proof. Take $(\underline{M}, \underline{t}) \in \mathcal{W}_r(\mathcal{T}_{\underline{H}})$. For every $\underline{P} = (X, P) \in \mathcal{T}_{\underline{H}}$, we have a morphism

$$\begin{array}{ccc} \underline{P} \otimes \underline{M} & \xrightarrow{t_{\underline{P}}} & \underline{M} \otimes \underline{P} \\ X \times V & \xleftarrow{s_X} & V \times X \\ P_x \otimes M_{v\cdot f(x)} & \xrightarrow{t_{P,v,x}} & M_v \otimes P_x, \end{array}$$

with $s_X(v, x) = (x, v \cdot f(x))$. Since the forgetful functor $\mathcal{T}_k \to \underline{\underline{\text{Sets}}}^{\text{op}}$ is strongly monoidal, it follows that $(V, s) \in \mathcal{W}_l(\mathcal{S}^G)$, so V is a right crossed G-module, by Proposition 4.3. Look at $t_{\underline{H}}$, and consider the composition

$$\underline{\rho} = t_{\underline{H}} \circ (\underline{\eta} \otimes \underline{M}),$$

namely

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\rho} & \underline{M} \otimes \underline{H} \\ V & \longleftarrow & V \times G \\ M_{v\cdot g} & \xrightarrow{\rho_{v,g}} & M_v \otimes H_g \end{array}$$

We will use the following Sweedler-type notation:

$$\rho_{v,g}(m) = m_{[0,v]} \otimes m_{[1,g]}.$$

step 1. t is completely determined by ρ.
For each $x \in X$, we fix $p_x \in P_x$, and consider the following morphism in $\mathcal{T}_{\underline{H}}$:

$$\begin{array}{ccc} \underline{H} & \xrightarrow{\varphi} & \underline{P} \\ G & \xleftarrow{f} & X \\ H_{f(x)} & \xrightarrow{\varphi_x} & P_x, \end{array}$$

with $\varphi_x(h) = p_x h$, for all $h \in H_{f(x)}$. From the naturality of $\underline{t}$, it follows that the following diagram is commutative:

$$\begin{tikzcd} \underline{H} \otimes \underline{M} \arrow[r, "t_{\underline{H}}"] \arrow[d, "\underline{f} \otimes \underline{V}"'] & \underline{M} \otimes \underline{H} \arrow[d, "\underline{V} \otimes \underline{f}"] \\ \underline{P} \otimes \underline{M} \arrow[r, "t_{\underline{P}}"] & \underline{M} \otimes \underline{P}. \end{tikzcd}$$

In particular, we have a commutative diagram, for all $v \in V$ and $x \in X$:

$$\begin{tikzcd} H_{f(x)} \otimes M_{v\cdot f(x)} \arrow[r, "t_{\underline{H},v,f(x)}"] \arrow[d, "\varphi_x \otimes M_{v\cdot f(x)}"'] & M_v \otimes H_{f(x)} \arrow[d, "M_v \otimes \varphi_x"] \\ P_x \otimes M_{v\cdot f(x)} \arrow[r, "t_{\underline{P},v,x}"] & M_v \otimes P_x \end{tikzcd}$$

and it follows that

$$t_{\underline{P},v,x}(p_x \otimes m) = m_{[0,v]} \otimes p_x m_{[1,f(x)]}. \tag{18}$$

step 2. From the definition of the right center, it follows that the following diagram commutes:

$$\begin{tikzcd} \underline{H} \otimes \underline{H} \otimes \underline{M} \arrow[r, "t_{\underline{H}\otimes\underline{H}}"] & \underline{M} \otimes \underline{H} \otimes \underline{H} \\ & \underline{H} \otimes \underline{M} \otimes \underline{H} & \end{tikzcd}$$

It follows that we have the following commutative diagram, for all $g, h \in G$ and $v \in V$:

$$\begin{tikzcd} H_g \otimes H_h \otimes M_{v\cdot(gh)} \arrow[r, "t_{\underline{H}\otimes\underline{H},v,gh}"] & M_v \otimes H_g \otimes H_h \\ & H_g \otimes M_{v\cdot g} \otimes H_h & \end{tikzcd}$$

Evaluating this diagram at $1_g \otimes 1_h \otimes m$, with $m \in M_{v\cdot(gh)}$, we find, using (18)

$$m_{[0,v]} \otimes \Delta_{g,h}(m_{[1,gh]}) = \rho_{v,g}(m_{[0,v]}) \otimes m_{[1,gh]},$$

so $\underline{\rho}$ is coassociative.

step 3. It follows from the definition of the left center that $t_{\underline{k}} = \underline{M} : \underline{k} \otimes \underline{M} \to \underline{M} \otimes \underline{k}$. Using (18), we compute, for all $m \in M_v$ that

$$m = t_{\underline{k},v,*}(m) = m_{[0,v]} \otimes 1 \cdot m_{[1,1]} = m_{[0,v]}\varepsilon(m_{[1,1]}),$$

and the counit property of $\underline{\rho}$ follows. Thus $\underline{\rho}$ defines a right $\underline{H}$-comodule structure on $\underline{M}$.

step 3. $\underline{t}_{\underline{H}}$ is right $\underline{H}$-linear, hence the following diagram commutes:

$$\begin{tikzcd}
\underline{H}\otimes\underline{M}\otimes\underline{H} \arrow[r] \arrow[d, "\underline{t}_{\underline{H}}\otimes\underline{H}"'] & \underline{H}\otimes\underline{M} \arrow[d, "\underline{t}_{\underline{H}}"] \\
\underline{M}\otimes\underline{H}\otimes\underline{H} \arrow[r] & \underline{M}\otimes\underline{H}
\end{tikzcd}$$

For all $v \in V$ and $g \in G$, we have the following commutative diagram, keeping in mind that $g\nu(v\cdot g) = \nu(v)g$:

$$\begin{tikzcd}
H_g\otimes M_{v\cdot g}\otimes H_{g\nu(v\cdot g)} \arrow[r] \arrow[d, "t_{\overline{H},v,g}\otimes H_{g\nu(v\cdot g)}"'] & H_g\otimes M_{v\cdot g} \arrow[d, "t_{\overline{H},v,g}"] \\
M_v\otimes H_g\otimes H_{\nu(g)g} \arrow[r] & M_v\otimes H_g
\end{tikzcd}$$

Evaluating this diagram at $1_g \otimes m \otimes h$, with $m \in M_{v,g}$, $h \in H_{g\nu(v\cdot g))}$, we find that (17) holds. Hence it follows that $\underline{M}$ is a right-right Yetter-Drinfeld module.

Conversely, let $\underline{M}$ be a right-right Yetter-Drinfeld module. We define

$$\underline{t}:\ -\otimes\underline{M}\to\underline{M}\otimes -$$

using (18). Standard computations show that $(\underline{M},\underline{t}) \in \mathcal{W}_l(\mathcal{T}_{\underline{H}})$. $\square$

In a similar way, we can compute the right weak center of ${}_{\underline{H}}\mathcal{T}$, and this is isomorphic to the category of left-right Yetter-Drinfeld modules ${}_{\underline{H}}\mathcal{YD}^{\underline{H}}$. Objects in ${}_{\underline{H}}\mathcal{YD}^{\underline{H}}$ are Turaev k-modules $\underline{M}$ with a right $\underline{H}$-comodule structure $\underline{\rho}:\ \underline{M}\to\underline{M}\otimes\underline{H}$ and a left $\underline{H}$-module structure $\underline{\psi}:\ \underline{H}\otimes\underline{M}\to\underline{M}$ such that

$$(\underline{\psi}\otimes\underline{\mu})\circ(\underline{H}\otimes\underline{\tau}\otimes\underline{H})\circ(\underline{\Delta}\otimes\underline{\rho}) = (\underline{M}\otimes\underline{\mu})\circ(\underline{\rho}\otimes\underline{H})\circ\underline{\tau}\circ(\underline{H}\otimes\underline{\psi})\circ(\underline{\Delta}\otimes\underline{M}).$$

Example 4.6. Let G be a group, fix $g \in G$, and let V be the orbit of g under the adjoint action $h \triangleleft k = k^{-1}hk$:

$$V = \{k^{-1}gk \mid k \in G\}.$$

Let $\nu:\ V \otimes G$ be the embedding of V into G. Then (V,ν), with the adjoint action, is a right crossed G-set. Assume that $\underline{H}$ is a crossed semi-Hopf G-coalgebra. Recall from [5, 10, 11] that this means that $\underline{H}$ is a semi-Hopf G-coalgebra with a family of algebra isomorphisms

$$\varphi_h^k:\ H_k \to H_{hkh^{-1}},$$

satisfying the following conditions (we omit the upper index if no confusion is possible):

$$\varphi_h\circ\varphi_k = \varphi_{kh}\ ;\ \varphi_1^k = H_k;$$
$$(\varphi_k\otimes\varphi_k)\circ\Delta_{l,h} = \Delta_{klk^{-1},khk^{-1}}\ ;\ \varepsilon\circ\varphi_g^1 = \varepsilon.$$

We also assume that the following additional condition holds:

$$\varphi_h^l = \varphi_k^l \text{ if } hlh^{-1} = klk^{-1}. \tag{19}$$

Take $M \in {}_{H_g}\mathcal{M}$. For each $v = h^{-1}gh \in V$, let $M_v = M$ as a k-module, with left H_v-action $a\cdot m = \varphi_h^v(a)m$, for all $a \in H$. This is well-defined, because of (19), and $\underline{M} = (V,(M_v)_{v\in V}) \in {}_{\underline{H}}\mathcal{T}$.

Assume now that we have a family of maps $\rho_l: M \to M \otimes H_l$, indexed by $l \in G$, as in Example 3.3. Then we have the following right $\underline{H}$-coaction on $\underline{M}$:

$$\rho_{v,k} = \rho_k : M_{k^{-1}vk} = M \to M_v \otimes H_k = M \otimes H_k.$$

We use the Sweedler notation

$$\rho_l(m) = m_{[0]} \otimes m_{[1,l]}.$$

With this action and coaction, $\underline{M} \in {}_{\underline{H}}\mathcal{YD}^{\underline{H}}$ if and only if

$$\begin{aligned} &\varphi_h(a_{(1,h^{-1}gh)})m_{[0]} \otimes a_{(2,l)}m_{[1,l]} \\ &\quad = \big(\varphi_{hl}(a_{(2,l^{-1}h^{-1}ghl)})m\big)_{[0]} \otimes \big(\varphi_{hl}(a_{(2,l^{-1}h^{-1}ghl)})m\big)_{[1,l]}a_{(1,l)}, \end{aligned}$$

for all $h, l \in G$ and $a \in H_{h^{-1}ghl}$. Taking $h = 1$, we find

$$\begin{aligned} &a_{(1,g)})m_{[0]} \otimes a_{(2,l)}m_{[1,l]} \\ &\quad = \big(\varphi_l(a_{(2,l^{-1}gl)})m\big)_{[0]} \otimes \big(\varphi_l(a_{(2,l^{-1}gl)})m\big)_{[1,l]}a_{(1,l)}, \end{aligned} \tag{20}$$

for all $l \in G$ and $a \in H_{gl}$.
Recall the definition of g-Yetter-Drinfeld module, introduced by Zunino in [11]. Let $\underline{H}$ be a crossed Hopf G-coalgebra, and fix $g \in G$ and $M \in {}_{H_g}\mathcal{M}$. Assume that we have maps ρ_l as above. Then M is a g-Yetter-Drinfeld module if the following compatibility conditions are satisfied, for all $l \in G$ and $a \in H_{gl}$:

$$\begin{aligned} &a_{(1,g)})m_{[0]} \otimes a_{(2,l)}m_{[1,l]} \\ &\quad = (a_{(2,g)}m)_{[0]} \otimes (a_{(2,g)}m)_{[1,l]}\varphi_{\alpha^{-1}}(a_{(1,glg^{-1})}). \end{aligned} \tag{21}$$

Observe that the conditions (20) and (21) are not the same.

5. Further generalizations

5.1. In the definition of the Turaev and Zunino categories, we can replace the category of k-modules by any monoidal category $\mathcal{C}$. The Turaev category $\mathcal{T}_{\mathcal{C}}$ and the Zunino category $\mathcal{Z}_{\mathcal{C}}$ will be symmetric (resp. braided) if $\mathcal{C}$ is symmetric (resp. braided).

5.2. The category $\underline{\underline{\text{Sets}}}$ can be replaced by any full subcategory of $\underline{\underline{\text{Sets}}}$ containing a singleton and closed under finite cartesian products, for example the category of finite sets, or the category of countable sets.

5.3. (E. Villanueva) Take $\underline{M} = (X, (M_x)_{x \in X}) \in \mathcal{T}_k$, and consider X as a topological space with the discrete topology. $\underline{M}$ can then be viewed as a (pre)sheaf of k-modules on X. Let $\mathcal{S}_k$ be the category with objects $(X, \mathcal{F})$, with X a topological space, and $\mathcal{F}$ a sheaf of k-modules on X. Then $\mathcal{T}_k$ is a full subcategory of $\mathcal{S}_k$.

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